Mathematics 24 Midterm 1 Take-Home Part
Spring 2013
Due Wednesday, April 24 in class

1. (20 points) Let V be a vector space, let $S \subseteq V$ be a subset and let $\langle S \rangle$ be the subspace spanned by S. Prove

$$\langle S \rangle = \bigcap_{S \subseteq W} W,$$

where the intersection is of all subspaces W which contain S. (You may assume without proof that the right-hand side is a vector space.)

(a) $S \subseteq \bigcap_{S \subseteq W} W$: $\bigcap_{S \subseteq W} W$ is a subspace of V

(Thm. 1.4) $S \subseteq W$ $\therefore S \subseteq \bigcap_{S \subseteq W} W$. Since $\langle S \rangle$

is the smallest ~~subspa~~ space containing S,

$\langle S \rangle \subseteq \bigcap_{W \supseteq S} W$

(b) $\bigcap_{S \subseteq W} W \subseteq \langle S \rangle$: $\bigcap_{S \subseteq W} W$ is the intersection

of <u>all</u> subspaces containing S. $\langle S \rangle$ is a

subspace containing S $\therefore \bigcap_{S \subseteq W} W \subseteq \langle S \rangle$

$\therefore \langle S \rangle = \bigcap_{S \subseteq W} W$

2. (15 points) Let $T : V \to V$ be a linear transformation and define $T^2 : V \to V$ by $T^2(v) = T(T(v))$ for all $v \in V$. Assume that $T^2 = T$ and prove

$$V = N(T) \oplus W,$$

where W is the subspace defined by

$$W = \{v \in V \mid T(v) = v\}.$$

See the second definition on p. 22 for the two conditions for a direct sum. Hint: consider $v - T(v)$.

(a) $NT \cap W = \{0\}$: Suppose $v \in NT$, $v \in W$

$0 = Tv = v \quad \therefore v = 0$

(b) $NT + W = V$: $v \in V \quad T(v - Tv) =$

$Tv - T^2 v = Tv - Tv = 0 \quad \therefore v - Tv \in NT$

Say $v - Tv = u$ for $u \in NT$, so $v = u + Tv$. Then

$Tv \in W$ since $T(Tv) = T^2 v = Tv$

$\therefore v = u + Tv \in NT + W$ so $NT + W = V$.

$\therefore V = NT \oplus W$.

3. (20 points) Let V be a vector space and $T : V \to V$ a linear transformation. Prove

1. If $V = R(T) + N(T)$, then $V = R(T) \oplus N(T)$.
2. If $R(T) \cap N(T) = \{0\}$, then $V = R(T) \oplus N(T)$.

Let $n = \dim V$, $r = \dim RT$, $k = \dim NT$

1. Assume $V = RT + NT$. Then

$n = \dim(RT + NT) = r + k - \dim(RT \cap NT)$

(p. 57, 29(a)). But $n = r + k$ by Thm. 2-3

$\therefore \dim(RT \cap NT) = 0 \quad \therefore RT \cap NT = \{0\}$

$\therefore V = RT \oplus NT$

2. ~~with~~ $\dim(RT + NT) = r + k - \dim(RT \cap NT) = r + k$

since $RT \cap NT = \{0\}$ But $n = \dim V = r + k$

But $RT + NT \subseteq V$ and $\dim(RT + NT) = \dim V$

$\therefore RT + NT = V \quad \therefore V = RT \oplus NT$